1. T is a mapping defined by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x-y \\ x+y \\ x \end{bmatrix}$$

(a) [2] What is the domain and codomain of T? $T: \mathbb{R}^2 \to \mathbb{R}^3$

(b) [3] Is T linear?

(c) [3] Find the standard matrix of T if possible.

(d) [2] Use the standard matrix to find

$$T\left(\begin{bmatrix} 5 \\ 4 \end{bmatrix}\right)$$

(b) $T(\vec{u}+\vec{v}) = T\left(\begin{bmatrix} u_1+v_1 \\ u_2+v_2 \end{bmatrix}\right) = \begin{pmatrix} u_1+v_1-u_2-v_2 \\ u_1+v_1+u_2+v_2 \\ u_1+v_1 \end{pmatrix}$

$= \begin{pmatrix} u_1-u_2 \\ u_1+u_2 \\ u_1 \end{pmatrix} + \begin{pmatrix} v_1-v_2 \\ v_1+v_2 \\ v_1 \end{pmatrix} = T(\vec{u}) + T(\vec{v})$

$T(k\vec{u}) = T\left(\begin{bmatrix} ku_1 \\ ku_2 \end{bmatrix}\right) = \begin{pmatrix} ku_1-ku_2 \\ ku_1+ku_2 \\ ku_1 \end{pmatrix} =$

$= k\begin{pmatrix} u_1+u_2 \\ u_1+u_2 \\ u_1 \end{pmatrix} = k\,T(\vec{u})$

$\to$ linear

c) $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}$

d) $T\left(\begin{bmatrix} 5 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 9 \\ 5 \end{bmatrix}$

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2. Let

$$A = \begin{bmatrix} 1 & -1 & 3 \\ 0 & 2 & 4 \\ 3 & -1 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & -1 & 3 & 1 \\ 0 & 2 & 4 & 0 \\ 3 & -1 & 0 & 0 \end{bmatrix},$$

Perform the operation or explain why it is not defined.

(a) [2] $A + B$. ~~[illegible]~~ $A \in \mathbb{R}^{3\times 3}$, $B \in \mathbb{R}^{3\times 4}$ $A + B$ not defined

(b) [3] AB.

$$AB = \begin{bmatrix} 10 & -6 & -1 & 1 \\ 12 & 0 & 8 & 0 \\ 3 & -5 & 5 & 3 \end{bmatrix}$$

3. (a) [6] Write parametric and normal equations of the plane that passes through the origin and contains the line L given by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \overset{P_0}{\begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}} + t \overset{\vec{v}}{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}$$

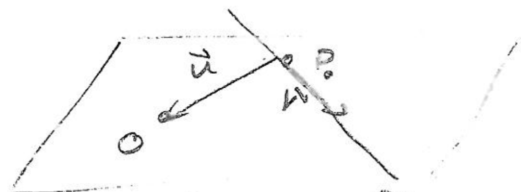
(b) [5] Determine whether the line L given by

$$\frac{x}{2} = \frac{y-2}{3} = \frac{z+1}{1}$$

is parallel to the plane P given by the equation 2x +4y-z=0.

If they are parallel, find the distance between them, if they are not parallel find the point of intersection.

a)



$$\vec{u} \quad \overrightarrow{P_0O} = \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix}$$

$$P: \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + s \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix}, \quad s, t \in \mathbb{R}$$

$$\vec{N} = \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 3 \\ -1 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 6 \\ -6 \\ 2 \end{bmatrix}$$

$$P: 6(x-1) - 6(y-0) + 2(z-?) = 0$$

b) $\vec{v} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$ L, P parallel iff $\vec{v} \perp \vec{N}$

$$\vec{v} \cdot \vec{N} = 12 - 18 + 2 = -4 \neq 0 \text{ not } \perp$$

$$Q_0 = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + s \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} + k \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} -1 & 1 & -2 & -1 \\ 0 & 2 & -3 & 2 \\ 3 & 3 & -1 & -4 \end{array}\right] \xrightarrow{R_3 + 3R_1} \left[\begin{array}{ccc|c} -1 & \overset{4}{1} & -2 & -1 \\ 0 & 2 & -3 & -2 \\ 0 & 6 & -7 & -7 \end{array}\right] \xrightarrow{R_3 - 3R_2} \left[\begin{array}{ccc|c} -1 & 1 & -2 & -1 \\ 0 & 2 & -3 & -2 \\ 0 & 0 & 2 & -1 \end{array}\right]$$

$$k = -\frac{1}{2} \quad I = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} + -\left(\frac{1}{2}\right) \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1/2 \\ -3/2 \end{bmatrix}$$

4. [4] Find all the values of the parameter a such that the following linear system (i) is inconsistent; (ii) has infinitely many solutions; (iii) has exactly one solution.

$$\begin{aligned} 2x - 7y + 3z &= 0 \\ x - 5y + 4z &= -3 \\ -2x + y + az &= 0 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 2 & -7 & 3 & 0 \\ 1 & -5 & 4 & -3 \\ -2 & 1 & a & a \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cccc} 1 & -5 & 4 & -3 \\ 2 & -7 & 3 & 0 \\ -2 & 1 & a & 0 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & -5 & 4 & -3 \\ 0 & 3 & -5 & 6 \\ 0 & -9 & 8+a & 6 \end{array}\right] \rightarrow \left[\begin{array}{cccc} 1 & -5 & 4 & -3 \\ 0 & 3 & -5 & 6 \\ 0 & 0 & -7+a & 24 \end{array}\right]$$

(i) no solution if $-7+a=0 \Longleftrightarrow a=7$

(ii) $24 \neq 0$ has never infinitely many solutions

(iii) $a \neq 7$ has exactly one solution

5. ~~(a)~~ i. [2] Give the definition of a linear mapping (transformation) corresponding to a matrix $A \in \mathbb{R}^{m \times n}$.

ii. [2] True or False? (include a short explanation) A system is inconsistent if the number of equations exceeds the number of unknowns (variables) of the system.

iii. [2] True or False? (include a short explanation) The number of pivots in the REF of a matrix A gives the number of linear independent column vectors in A.

i) $\cancel{F_A}$: Let $A \in \mathbb{R}^{m \times n}$ then $F_A : \mathbb{R}^n \to \mathbb{R}^m$ with $F_A(\vec{x}) = A\vec{x}$ is a linear mapping corresponding to $A \in \mathbb{R}^{n \times m}$.

ii) False,

$$\begin{aligned} 2x + y &= 0 \\ 2x + y &= 0 \\ 2x + y &= 0 \end{aligned}$$

is not inconsistent.

iii) True

Vectors are linear independent, if the homogeneous system only has the trivial solution. The system only including the columns including the pivots has only the trivial solution.

$\to$ True